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POSITIVE SOLUTIONS FOR A CONCAVE SEMIPOSITONE DIRICHLET PROBLEM†

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1. INTRODUCTION

We study the boundary value problem

$$-\Delta u(x) = \lambda f(u), \qquad x \in \Omega; \tag{1.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega; \tag{1.2}$$

where Δ is the Laplacian operator, Ω is the unit ball in $\mathbb{R}^N$, $N \geq 1$, $f\colon [0, \infty) \to \mathbb{R}$ is a concave smooth function such that

$$f(0) < 0. \tag{1.3}$$

Nonnegative solutions to (1.1)–(1.2) are known to be positive and radial (see [1]). Also for (1.1)–(1.2) to have a positive solution it is necessary and sufficient that

$$\int_0^\alpha f(s)\,\mathrm{d}s > 0 \qquad \text{for some } \alpha > 0. \tag{1.4}$$

In fact, the results of [2] and [3] show that for any smooth bounded region the equations (1.1)–(1.2) have positive solutions for $\lambda > 0$ large if (1.4) holds and

$$\lim_{u\to\infty} f'(u) \leq 0. \tag{1.5}$$

On the other hand, since positive solutions to (1.1)–(1.2) are radial then $v(r) = u(x)$ for $\|x\| = r$ satisfies

$$v'' + \frac{N-1}{r}v' + \lambda f(v) = 0, \quad r \in [0, 1] \qquad u'(0) = u(1) = 0. \tag{1.6}$$

Letting $F(t) = \int_0^t f(s)\,\mathrm{d}s$, we see that $E(r) \equiv (v'(r))^2/2 + F(v(r)))$ satisfies

$$E'(r) = -\frac{n}{r}(v'(r))^2 \leq 0. \tag{1.7}$$

Since $E(1) \geq 0$ we have that either $E(0) = F(u(0)) > 0$ or v is a constant function. Since no constant function satisfies (1.6) we conclude that (1.4) is also a necessary condition for (1.1)–(1.2) to have a solution.

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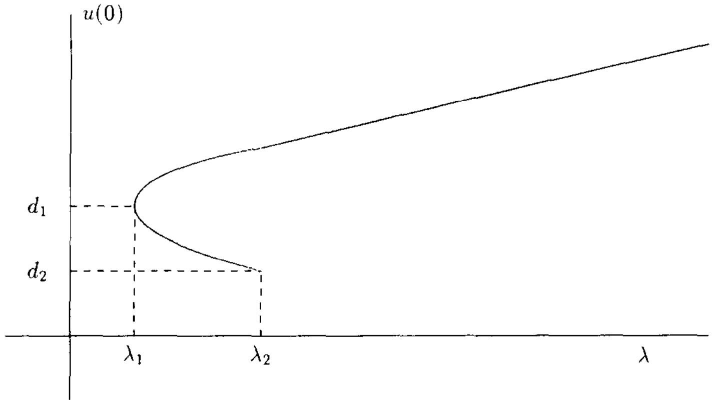

Fig. 1.

The rest of the article is devoted to showing that if (1.4)–(1.5) hold then the set of positive solutions to (1.1)–(1.2) corresponds to Fig. 1. That is, we prove the following result.

THEOREM 1.1. If (1.4)–(1.5) hold then the set of nonnegative solutions to (1.1)–(1.2) is connected and, for each $\lambda > 0$ it has at most one stable solution and one unstable solution. There exist positive numbers $0 < \lambda_1 < \lambda_2$ such that: (1.1)–(1.2) has a stable solution if and only if $\lambda > \lambda_1$, and (1.1)–(1.2) has an unstable solution if and only if $\lambda \in [\lambda_1, \lambda_2]$.

Since Theorem 1.1 was proven in [4] when $\lim_{u\to\infty} f'(0) = 0$ we restrict ourselves in this paper to the case where f has a second positive zero. That is, we assume f to be like in Fig. 2, where $F(\theta) = 0$ and $f'(\tau) = 0$. We note that without loss of generality we can assume that $\theta > \tau$. A key piece in the proof of Theorem 1.1 is showing that, for $\lambda > 0$ large, (1.1)–(1.2) has a unique positive solution. When $\lim_{u\to\infty} f'(u) = 0$, this proof was given in [5].

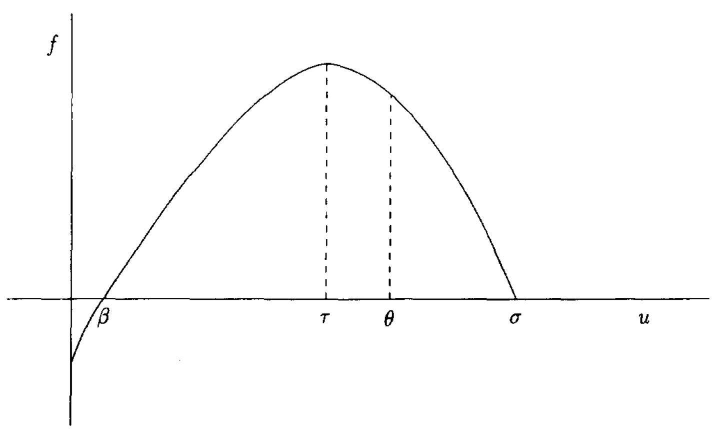

Fig. 2.

In this case it is shown that solutions to (1.1)–(1.2) satisfy $|u'(r)| = O(\lambda)$ for r close to 1, which is not valid when $\lim_{u\to\infty} f'(u) < 0$. In the latter case $|u'(r)| \le O(\lambda^{1/2})$ for all $r \in [0, 1]$.

Equations such as (1.1)–(1.2) arise, for example, in population models in the presence of "harvesting".

2. *A PRIORI* ESTIMATES

The arguments of this, and the next, section hold under assumptions weaker than concavity. For the sake of simplicity we assume that

$$f' > 0 \quad \text{on } [0, \tau) \qquad \text{and} \qquad f' < 0 \quad \text{on } (\tau, \sigma]. \tag{2.1}$$

Let (λ, u) be a positive solution to (1.1)–(1.2). Since u is radial we have

$$u'' + \frac{N-1}{r} u' + \lambda f(u) = 0, \quad r \in [0, 1] \qquad u'(0) = u(1) = 0. \tag{2.2}$$

We now prove the following.

LEMMA 2.1. There exist positive constants c and $\bar{\lambda}$ such that if u is a positive solution to (2.2) then $(u'(r))^2 \ge c\lambda$ if $u(r) \le \tau$ and $\lambda > \bar{\lambda}$.

Since u is positive $u(0) > \beta$. Let $r_0 \in (0, 1)$ be such that $u(r_0) = \beta/2$ and $u(r) \le \beta/2$ for all $r \in [r_0, 1]$. Multiplying the second order differential equation in (2.2) by r^{N-1} and integrating on $[r, 1]$ we have

$$r^{N-1}u'(r) = u'(1) + \lambda \int_r^1 s^{N-1} f(u(s))\,ds \le \frac{\lambda f(\beta/2)}{N}(1 - r^N) \tag{2.3}$$

for $r \in [r_0, 1]$. Since $r \le 1$ we have $u'(r) \le \lambda f(\beta/2)(1 - r)$. Integrating on $[r_0, 1]$ we have $-\beta \le \lambda f(\beta/2)(1 - r_0)^2$. Thus

$$(1 - r_0)^2 \le \frac{-\beta}{\lambda f(\beta/2)}. \tag{2.4}$$

Therefore, there exists $r_1 \in [r_0, 1]$ such that

$$u'(r_1) \le -(-\beta\lambda f(\beta/2))^{1/2}/2 \equiv -c_1\lambda^{1/2}. \tag{2.5}$$

Let $r_2 < r_1$ be such that $u(r_2) = \beta$. Arguing as above for $r \in [r_2, r_1]$ we have

$$r^{N-1}u'(r) = r_1^{N-1}u'(r_1) + \lambda \int_r^{r_1} s^{N-1} f(u(s))\,ds \le -r_1^{N-1}c_1\lambda^{1/2}. \tag{2.6}$$

Integration of the latter inequality on $[r_2, r_1]$ yields

$$-\beta \le -r_1^{N-1}c_1\lambda^{1/2} \int_{r_2}^{r_1} s^{1-N}\,ds \le -c_1\lambda^{1/2}(r_1 - r_2). \tag{2.7}$$

Hence

$$r_1 - r_2 \le \frac{\beta}{c_1\lambda^{1/2}}. \tag{2.8}$$

Now replacing in (2.6) r_1 by r_2, using (2.6) and that f attains its maximum value at τ we conclude

$$r^{N-1}u'(r) = r_2^{N-1}u'(r_2) + \lambda \int_r^{r_2} s^{N-1}f(u(s))\,ds$$

$$\leq -r_2^{N-1}c_1\lambda^{1/2} + \frac{\lambda f(\tau)}{N}(r_2^N - r^N), \tag{2.9}$$

for $r \leq r_2$. Since $(r_2^N - r^N) \leq Nr_2^{N-1}(r_2 - r)$, from (2.9) we see that for $r_2 - r \leq c_1/2f(\tau)\lambda^{1/2}$

$$u'(r) \leq -(c_1/2)\lambda^{1/2}. \tag{2.10}$$

Let $r_3 \equiv r_2 - c_1/2f(\tau)\lambda^{1/2}$. Integrating on $[r_3, r_2]$ we have

$$\sigma > u(r_3) \geq \beta + \frac{c_1^2}{4f(\tau)}. \tag{2.11}$$

Since (1.5) implies that f is bounded, and since $E(r) \leq E(0) = \lambda F(u(0))$ for all $r \in [0, 1]$, we have

$$|u'(r)| \leq c_2\lambda^{1/2} \qquad \text{for all } r \in [0, 1]. \tag{2.12}$$

Let $\tau_1 \geq (\theta + \sigma)/2 > \tau > \tau_0 > \beta$ be such that $f(\tau_0) = f(\tau_1) = \min\{f((\theta + \sigma)/2), f(\beta + c_1^2/4f(\tau))\}$. Let $0 < r_5 < r_4 < 1$ be such that $u(r_5) = \tau_1$ and $u(r_4) = \tau_0$. Note that, by (2.11), $r_4 > r_3$. Since $f(u(s)) \geq f(\tau_0)$ for all $s \in [r_5, r_4]$, we have

$$r_4^{N-1}u'(r_4) = r_5^{N-1}u'(r_5) - \lambda \int_{r_5}^{r_4} s^{N-1}f(u(s))\,ds$$

$$\leq -\frac{f(\tau_0)\lambda}{N}(r_4 - r_5)(r_4^{N-1} + \cdots + r_5^{N-1}).$$

This and (2.12) imply

$$r_5 \geq r_4 - \frac{c_2N}{\lambda^{1/2}f(\tau_0)}. \tag{2.13}$$

Now using (2.4), (2.8), (2.13), $r_1 \geq r_0$, $r_4 > r_3$ we conclude

$$\begin{aligned} r_5 &\geq r_3 - \frac{c_2N}{\lambda^{1/2}f(\tau_0)} \\ &= r_2 - \left(\frac{c_1}{2f(\tau)} + \frac{c_2N}{f(\tau_0)}\right)\lambda^{-1/2} \\ &\geq r_1 - \left(\frac{\beta}{c_1} + \frac{c_1}{2f(\tau)} + \frac{c_2N}{f(\tau_0)}\right)\lambda^{-1/2} \\ &\geq 1 - \left(\frac{\beta}{c_1} + \frac{c_1}{2f(\tau)} + \frac{c_2N}{f(\tau_0)} + \left(\frac{-\beta}{f(\beta/2)}\right)^{1/2}\right)\lambda^{-1/2} \\ &\equiv 1 - c_3\lambda^{-1/2}. \end{aligned} \tag{2.14}$$

Finally we estimate $|u'(r)|$ for $u(r) \le \tau$. From the definition of E, for λ sufficiently large and $u(r) \le \tau$, we have

$$(u'(r))^2 = (u'(r_5))^2 + 2\lambda(F(u(r_5)) - F(u(r))) - 2\int_{r_5}^{r} \frac{n(u'(s))^2}{s}\, ds$$
$$\ge 2\lambda\left(F\left(\frac{\theta+\sigma}{2}\right) - F(\tau)\right) - \frac{(1-r_5)2n(c_2)^2\lambda}{r_5}$$
$$\ge \left(F\left(\frac{\theta+\sigma}{2}\right) - F(\tau)\right)\lambda \equiv c\lambda \tag{2.15}$$

where we have used (2.14) to estimate $1 - r_5$ and that $r \ge r_5$ for $u(r) \le \tau$.

3. UNIQUENESS OF POSITIVE SOLUTIONS FOR λ LARGE

By the maximum principle, and because $f(\sigma) = 0$, we see that if (1.1)–(1.2) has a positive solution than it has a maximal positive solution. Thus if we assume that (1.1)–(1.2) has two positive solutions u and v, without loss of generality, we can assume that $v(x) > u(x)$ for all $x \in \Omega$. We let $z \equiv v - u$, and $s_1, s_2, s_3 \in (0, 1)$ such that $u(s_1) = \tau_1$, $u(s_2) = \tau = v(s_3)$. Since u and v are radially decreasing we have $s_1 < s_2 < s_3$. By (2.1) and the definition of s_2 we have $f(v(s)) - f(u(s)) \le 0$ for all $s \in [0, s_2]$. Hence, letting $z \equiv v - u$ we have

$$t^{N-1}z'(t) = -\lambda\int_0^t s^{N-1}(f(v(s)) - f(u(s)))\, ds \ge 0, \tag{3.1}$$

for all $t \in [0, s_2]$. Since $z'(1) \le 0$ we have

$$\int_0^{s_1} s^{N-1}(f(u(s)) - f(v(s)))\, ds \le \int_{s_1}^{1} s^{N-1}(f(v(s)) - f(u(s)))\, ds.$$

This, (2.14), and (3.1) imply

$$k_1\int_0^{s_1} s^{N-1}(v(s) - u(s))\, ds \le \int_0^{s_1} s^{N-1}(f(u(s)) - f(v(s)))\, ds$$
$$\le \int_{s_1}^{1} s^{N-1}(f(v(s)) - f(u(s)))\, ds$$
$$\le c_3k_2\lambda^{-1/2}\max\{z(s); s \in [s_1, 1]\}$$
$$\le c_3k_2\lambda^{-1/2}\max\{z(s); s \in [s_2, 1]\}, \tag{3.2}$$

where $k_1 = \min\{-f'(t); t \in [\tau_1, \sigma]\}$, and $k_2 = \max\{|f'(t)|; t \in [0, \sigma]\}$. Now we are ready to prove the uniqueness of positive solutions for (1.1)–(1.2) when λ is large.

THEOREM 3.1. There exists $\lambda_0 > 0$ such that if $\lambda > \lambda_0$ equation (1.1)–(1.2) has a unique positive solution.

Proof. Recalling the Pohozaev identity (see [5]) we see that

$$(u'(1))^2 = 2\lambda \int_0^1 s^{N-1} P(u(s))\, ds,$$

where $P(t) = N\int_0^t f(s)\, ds - ((N-2)/2)tf(t)$. Similarly for $(v'(1))^2$. This, (2.15) and (3.2) give

$$\begin{aligned}
|z'(1)| &= \frac{2\lambda \int_0^1 s^{N-1}(P(v(s)) - P(u(s)))\, ds}{|u'(1) + v'(1)|} \\
&\leq \left(\frac{\lambda}{c}\right)^{1/2} k_3 \int_{s_1}^1 s^{N-1}\left(v(s) - u(s) + \frac{f(v(s)) - f(u(s))}{k_1}\right) ds \\
&\leq \left(\frac{\lambda}{c}\right)^{1/2} k_3 \int_{s_1}^1 s^{N-1}(v(s) - u(s))\left(1 + \frac{k_2}{k_1}\right) ds \\
&\leq \left(1 + \frac{k_2}{k_1}\right)\left(\frac{1}{c}\right)^{1/2} k_3 c_3 \max\{z(s); s \in [s_2, 1]\} \\
&\equiv k_4 \max\{z(s); s \in [s_2, 1]\},
\end{aligned} \tag{3.3}$$

where $k_3 = \max\{|P'(t)|;\ t \in [0, \sigma]\}$. Because $f(v(s)) - f(u(s)) > 0$ for $s \in [s_3, 1]$ we have

$$s^{N-1} z'(s) = z'(1) + \lambda \int_s^1 s^{N-1}(f(v(s)) - f(u(s))\, ds \geq z'(1). \tag{3.4}$$

Since for λ large $s_3^{N-1} > 1/2$ [see (2.14)], we have $z'(s) \geq -2k_4 \max\{z(s); s \in [s_2, 1]\}$ for all $s \in [s_3, 1]$. In particular, by (2.14) we have

$$z(t) \leq 2k_4 \max\{z(s); s \in [s_2, 1]\} c_3 \lambda^{-1/2}, \tag{3.5}$$

for all $t \in [s_3, 1]$. Since this implies that $u(s_3) \geq \tau - 2k_4 \max\{z(s); s \in [s_2, 1]\} c_3 \lambda^{-1/2}$, by (2.15) we have $s_2 \geq s_3 - 2k_4 \max\{z(s); s \in [s_2, 1]\} c_3 \lambda^{-1}/c^{1/2}$. Now for $t \in [s_2, s_3]$ we have

$$\begin{aligned}
t^{N-1} z'(t) &= s_3^{N-1} z'(s_3) + \lambda \int_t^{s_3} s^{N-1}(f(v(s)) - f(u(s)))\, ds \\
&\geq -\max\{z(s); s \in [s_2, 1]\}\left(2k_4 + \frac{(f(\tau) - f(0))2k_4 c_3}{c^{1/2}}\right).
\end{aligned} \tag{3.6}$$

Thus for λ sufficiently large we have

$$z'(t) \geq -2 \max\{z(s); s \in [s_2, 1]\}\left(2k_4 + \frac{(f(\tau) - f(0))2k_4 c_3}{c^{1/2}}\right),$$

for $t \in [s_2, s_3]$. Now for $t \in [s_2, 1]$ and λ sufficiently large, by (3.5) we have

$$\begin{aligned}
z(t) &\leq 2c_3 \lambda^{-1/2} \max\{z(s); s \in [s_2, 1]\}\left(2k_4 + \frac{(f(\tau) - f(0))2k_4 c_3}{c^{1/2}}\right) \\
&\leq (1/2) \max\{z(s); s \in [s_2, 1]\}.
\end{aligned} \tag{3.7}$$

Since (3.7) implies that $z \equiv 0$ on $[s_2, 1]$, by uniqueness of solutions to initial value problems we see that $v = u$, which concludes the proof. ■

4. PROOF OF THEOREM 1.1

Here we use extensively the developments of [4].

For $\lambda > 0$, $d > 0$, let $u(\cdot, \lambda, d)$ denote the solution to the initial value problem

$$u'' + \frac{N-1}{r}u' + \lambda f(u) = 0 \qquad r \in [0, 1],\ u'(0) = 0,\ u(0) = d. \tag{4.1}$$

Using the fact that f is concave and $\lim_{u\to\infty} f'(u) \leq 0$, it follows that the function $u(\cdot, \lambda, d)$ has the following properties:

(a) If $u(\cdot, \lambda, d) > 0$ in $[0, 1)$, $u(1, \lambda, d) = 0$, then $u_d(\cdot, \lambda, d) = 0$ has at most one zero in $[0, 1]$.

(b) If $u(\cdot, \lambda_1, d_1) > 0$ in $[0, 1)$, $u(1, \lambda_1, d_1) = 0$, and $u_d(1, \lambda_1, d_1) = 0$ then $u_\lambda(1, \lambda_1, d_1) < 0$ and $u_{dd}(1, \lambda_1, d_1) > 0$. Moreover, there exists $\varepsilon > 0$ and a differentiable function

$$\Lambda: (d_1 - \varepsilon, d_1 + \varepsilon) \to R$$

such that $u(1, \Lambda(d), d) = 0$, $\Lambda'(d_1) = 0$ and $\Lambda''(d_1) > 0\}$.

(c) If $\Gamma \subset R^2$ is a connected component of $\{(\lambda, d); u(\cdot, \lambda_0, d_0) > 0$ in $[0, 1)$, $u(1, \lambda_0, d_0) = 0$, then Γ is unbounded in the λ direction.

These properties were proved in Lemmas 1-3 in [4] in the case $\lim_{u\to\infty} f'(u) = 0$. However we point out that in Lemma 3 of [4] the proof misses the observation that for each $\mu > 0$ there exists an $M(\mu)$ such that $\|u\|_\infty \leq M(\mu)$ for every $\lambda \in (0, \mu)$, which follows from $\lim_{u\to\infty} f'(u) = 0$. In the case $\lim_{u\to\infty} f'(u) < 0$ it is easy to observe that $M(\mu) = \sigma$. Further since there are no positive solutions for $\lambda > 0$ small, we must have

(d) There exists $(\lambda_1, d_1) \in \Gamma$ such that $u_d(1, \lambda_1, d_1) = 0$.

Now from Theorem 3.1 and (c) we see that

$$\{(\lambda, d); u(\cdot, \lambda_0, d_0) > 0 \text{ in } [0, 1],\ u(1, \lambda_0, d_0) = 0\} \equiv \Gamma$$

is connected. From (b) and the implicit function theorem we see that Γ is a differentiable curve. Also, by the uniqueness of solutions to the initial value problem (4.1)

$$u(r/\rho, \lambda, d) = u(r, \lambda/\rho^2, d), \qquad \text{for any } \rho > 0. \tag{4.2}$$

Suppose that $(\alpha, d) \in \Gamma$ and $(\beta, d) \in \Gamma$ with $\alpha < \beta$. Using (4.2) we see that $u((\alpha/\beta)^{1/2}, \beta, d) = 0$. Hence for $\lambda = \beta$ equation (1.1)–(1.2) has a solution with interior zeros ($r = \alpha/\beta$). Because nonnegative solutions to (1.1)–(1.2) are positive we have a contradiction. Thus for each $d > 0$ there exists at most one $\lambda > 0$ such that $(\lambda, d) \in \Gamma$. Therefore the function Λ in (b) can be extended to $A = \{d; (\lambda, d) \in$ for some $\lambda > 0\}$. Also by (b) the number d_1 is unique, $\Lambda'(d) > 0$ for $d > d_1$, and $\Lambda'(d) < 0$ for $d < d_1$. Because nonnegative solutions to (1.1)–(1.2) satisfy $u(0) > \theta$, $d_2 = \inf A \geq \theta$. Since, by Theorem 3.1, the problem (1.1)–(1.2) has a unique solution for λ large, and because of results [2] for $\lambda > 0$ large this problem has a stable solution, we conclude that for $\lambda > 0$ large enough $u_d(1, \lambda, d) > 0$. On the other hand since $\Lambda'(d) < 0$, for $d \in (d_2, d_1)$, we have $u_d(1, \Lambda(d), d) < 0$, and by uniqueness for λ large we have $\lim_{d\to d_2} \Lambda(d) < \infty$. Further, by continuous dependence of solutions to (4.1) in parameters we see that $(\lim_{d\to d_2} \Lambda(d), d) \in \Gamma$. Taking $\lambda_2 = \lim_{d\to d_2} \Lambda(d)$ the theorem is proven. ■

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